Tomographic Reconstruction Using Ridge Functions

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Abstract – *In this paper tomographic reconstruction based on the concept of ridge functions (Logan and Shepp) is considered. A reconstruction approach for the ridge functions from a finite number of arbitrary projections is suggested within the framework of parallel beam geometry. The method deals with images that can be presented as a sum of ridge functions. We derive a formula to calculate the ridge functions from the set of arbitrary projections. In the case of few projections an approach for detection of ridge functions with unknown directions is suggested. Results of numerical simulation are presented.*

Keywords : ridge functions, reconstruction, tomography

1. INTRODUCTION

Let *f* be a square integrable function which vanishes outside the unit disk *D* in the plane *(x,y)*. The Radon transform R_θ maps a function onto its line integrals along some direction determined by an angle θ and a distance *s* from the origin [1]:

$$R_\theta f = \int f(s\cos\theta - t\sin\theta, s\sin\theta + t\cos\theta)dt$$

We will denote a single projection by $p(\theta,s)=R_\theta f(s)$. Theoretically the inverse Radon transform is given by

$$f(x,y) = \frac{1}{2\pi^2}\int_0^\pi\int_{-1}^1 \frac{\partial p(\theta,s)/\partial s}{x\cos\theta + y\sin\theta - s}dsd\theta.(1)$$

This formula supposes the total knowledge of *p* in $[0,\pi)\times[-1,1]$, thus an infinite number of projections. In practical tomographic reconstruction [2] only a limited number of projections is available. We represent the directions in the tuple $\omega=(\omega_1,...,\omega_n)$ and the

$$R_\omega f = (R_{\omega_1} f, ..., R_{\omega_n} f).$$

associated projections are denoted by

Reconstructing *f* from $R_\omega f$ in the case of equally spaced projections is mostly done by filtered backprojection [3]. In this technique the following approximation of (1) is used:

$$f_{FBP}(x,y) = \frac{1}{n}\sum_{i=1}^n r_i(x\cos\omega_i + y\sin\omega_i). \quad (2)$$

The function r_i is defined as

$$r_i(s) = \int p(\omega_i, t)k(s-t)dt. \qquad (3)$$

The convolution kernel *k* is a regularized version of the inverse Fourier transform of $|\rho|$.

2. RIDGE FUNCTIONS

Logan and Shepp suggested an alternative to the aforementioned convolution-based approach. They introduced ridge functions *h* associated with a direction θ which are of the form

$$h(x,y) = h(x\cos\theta + y\sin\theta).$$

The following theorem is due to Logan and Shepp [4]:

Theorem 1. *Let* $\omega=(\omega_1,..,\omega_n)$ *be a tuple of*

$$R_{\omega_i} f = R_{\omega_j} H, i = 1,...,n.$$

distinct angles. Let H be the unique function in $L^2(D)$ *of the smallest norm which satisfies*

Then there exist functions $h_1,\ldots h_n$ *such that*

$$H(x,y) = \sum_{i=1}^n h_i(x\cos\omega_i + y\sin\omega_i).$$

Theorem 1 states the existence of the ridge functions. This work reports on construction algorithms for these functions, given an arbitrary number of projections $R_\omega f$. The basis of the approach consists in the observation of Davison [5] and Louis [6] that the Chebyshev polynomials

$$\frac{1}{a} R_{\theta_i} R^{\#}_{\theta_j} U_k = \frac{2}{k} U_k(\cos(\theta_i - \theta_j)) U_k$$

U_k are eigenfunctions of the operator $a^{-1}RR^{\#}$:

Here $R_\theta^{\#}$ is the backprojection operator, $a(t)=(1-t^2)^{1/2}$. Using these results, the following theorem establishes an analytical formula which enables to calculate the ridge functions directly from the projections.

Theorem 2. *The ridge functions $h_i(s)$ from Theorem 1 have the following form:*

$$h_i(s) = \frac{1}{\pi} \sum_{k=0}^{\infty} \sum_{j=1}^{n} \eta_{ij}^{(k)} U_k(s) \int p(\omega_j, t) U_k(t) dt, \quad (4)$$

Where U_k are the Chebyshev polynomilas of the second kind, $\eta_{ij}^{(k)}$ are entries of the matrix Λ_k^{-1} (or Λ_k^{+} - generalized inverse), $\Lambda_k=(\lambda_{ij}^{(k)})$,

$$\lambda_{ij}^{(k)} = \frac{\sin(k(\omega_i - \omega_j))}{k \sin(\omega_i - \omega_j)}. \quad (5)$$

In the case of equally spaced angles the calculation can be reduced to a computational more efficient formula. If the angles ω_i are equally spaced ($\omega_i=\pi(i-1)/n$, $i=1,\dots,n$) then analytical inversion of the matrix Λ_k is possible [7]:

Theorem 3. *For $k < n$, matrix Λ_k is singular and its generalized inverse Λ^{+}_k is*

$$\Lambda_k^{+} = \frac{k^2}{n^2} \Lambda_{k}.$$

Theorem 4. *For $k > n$ and $k=mn+l$ with $l=0,\dots,n-1$ the matrix Λ_k is nonsingular and its inverse is*

$$\Lambda_k^{-1} = \frac{mn+l}{m(m+1)n^2}((2m+1)nI - (mn+1)\Lambda_l).$$

Relevant considerations can be found also in the paper [8].

2.1 Numerical experiments

For our numerical modeling we use the Shepp-Logan phantom [3] with integer grey level values of ellipses (Figure 1). The size of discretized image is 128x128.

Figure 1. Shepp and Logan phantom

We generate 128 equispaced projections each with 128 samples. For comparison with suggested approach the filtered backprojection (FBP) technique (2)-(3) is applied (Figure 2).

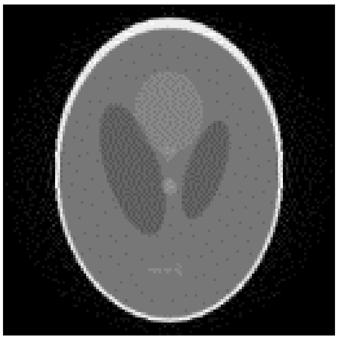

Figure 2. Reconstruction by filtered backprojection

The sum of 128 ridge functions is computed in accordance with formulae (4)-(5) , summation over k is cutted after 128 terms. This parameter of cutting is chosen empirically, optimal choice of truncating the series in (4) stays open. The result is presented in Figure 3.

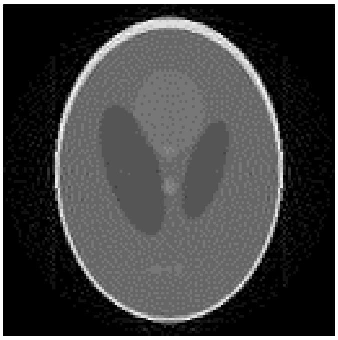

Figure 3. Reconstruction with ridge functions

We can see that compared with standard FBP method, the reconstuction with ridge functions sufferes from oscillations. Profiles of the 64-th column of test image, filtered backprojection and reconstruction with ridge functions (from left to right) are visualized in Figure 4.

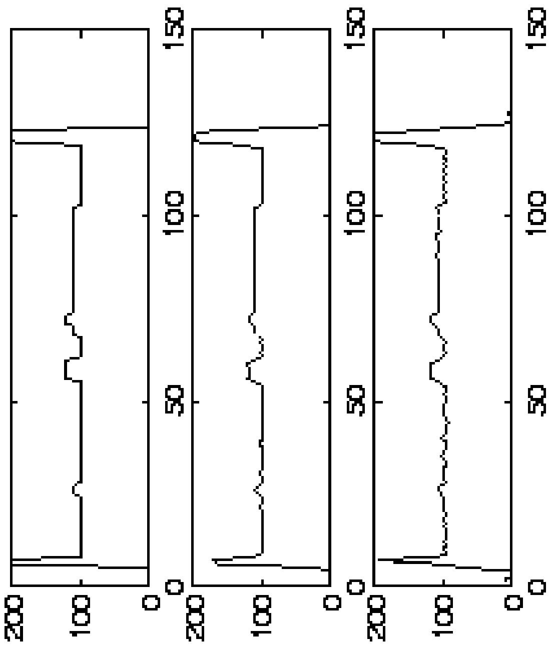

Figure 4. Profiles of the Shepp-Logan phantom and the two reconstructed images along the 64-th column.

3. RECONSTRUCTION OF FEW RIDGE FUNCTIONS WITH KNOWN DIRECTIONS

Let us assume that data acqusition system provides us with n projections of function f in directions $\omega=(\omega_1,\dots,\omega_n)$. Suppose *a priori* that our function f is a sum of n ridge functions h_i which directions $\alpha=(\alpha_1,\dots,\alpha_n)$ are known. We try to reconstruct f from projections $R_\omega f = (p(\omega_1,s),\dots,p(\omega_n,s)) = (p_1,\dots,p_n)$.

Generalizing the approach of previous section we can derive the following form of ridge functions $h_i(s)$ in terms of projection data $R_\omega f$, similar to (4):

$$h_{\alpha_i} = \frac{1}{\pi}\sum_{k=0}^{\infty}\sum_{j=1}^{n}\eta_{ij}^{(k)}U_k(s)\int p_{\omega_j}(t)U_k(t)dt, \quad (6)$$

Where $h_{ij}^{(k)}$ are entries of the matrix Λ_k^{+}, with elements of matrix Λ_k:

$$\lambda_{ij}^{(k)} = \frac{\sin(k(\omega_i-\alpha_j))}{k\sin(\omega_i-\alpha_j)}. \quad (7)$$

If projections are uniformly spaced in $[0,\pi)$ and number n is sufficiently large, reconstruction with ridge functions nearly coincides with FBP reconstruction, as we have seen in section 2. In this section the case of small number of projections is investigated numerically.

3.1 Numerical experiments

The phantom consists of five narrow ellipses with centers at the origin; function f is constructed as a sum of five approximately ridge functions with parameters $\alpha=(18^o, 54^o, 90^o, 126^o, 162^o)$. Axes of ellipses are $a=0.05$ and $b=0.99$ with ellipse's equation $x^2/a^2+y^2/b^2=1$ (Figure 5).

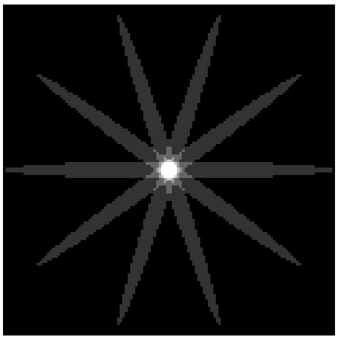

Figure 5. The test image

Densities for ellipses are chosen the same and equal 50 grey levels so that density of central light spot is 250. Angles under which five equispaced projections are generated constitutes 5-tuple $\omega = (0^o, 36^o, 72^o, 108^o, 144^o)$. Results of filtered backprojection are shown in Figure 6.

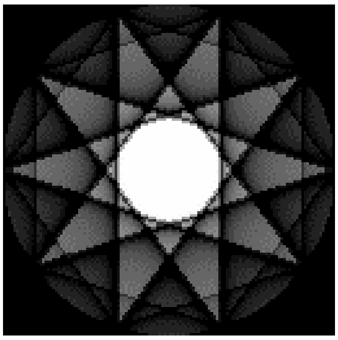

Figure 6. Reconstructed image using FBP

Shown in Figure 7 is reconstruction obtained from projections $R_\omega f$ using formulae (6)-(7).

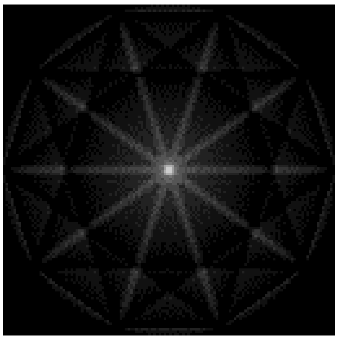

Figure 7. Reconstruction using ridge functions

We present also more complicated example which clearly shows limits of applicability of the approach considered. The test image is a sum of five ellipses some of which are not elongated structures (Figure 8). Angles of major axes inclinations are $\alpha = (0^o, 20^o, 45^o, 70^o, 90^o)$.

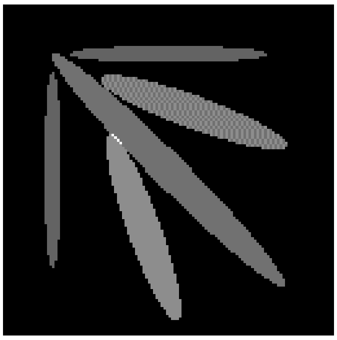

Figure 8. The test image

Projections are generated and backprojected (after standard filtration) under the angles $\omega = (0^o, 36^o, 72^o, 108^o, 144^o)$ (Figure 9).

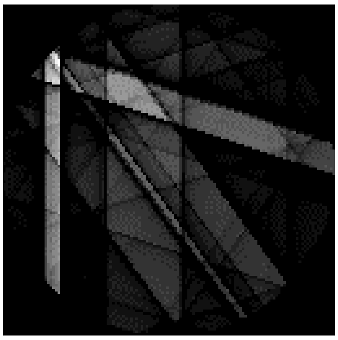

Figure 9. Reconstructed image using FBP

In Figure 10 is shown the alternative reconstruction based on *a priori* knowledge about the image structures directions α.

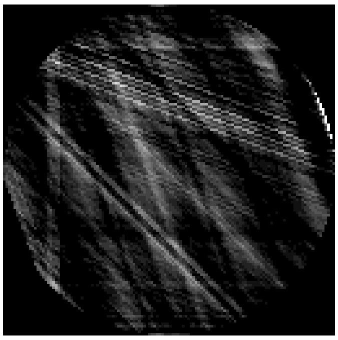

Figure 10. Reconstruction using ridge functions

Shown in Figure 11 are profiles of the phantom, FBP reconstruction and ridge functions reconstruction along the 54-th column.

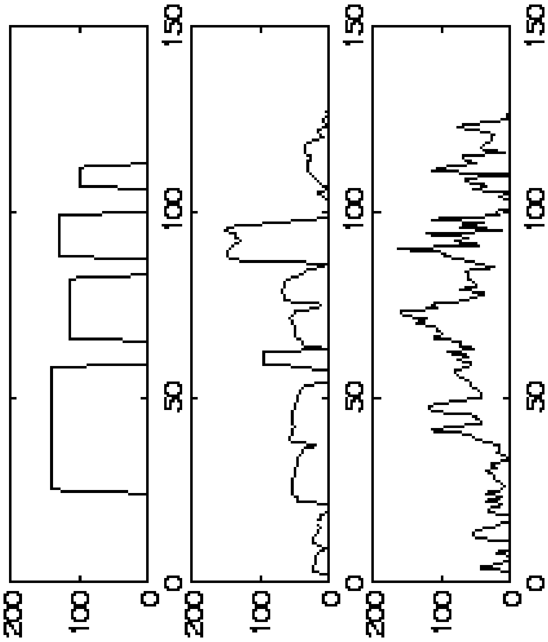

Figure 11. Profiles of the phantom and the reconstructed images along the 54-th column.

4. DETECTION OF DIRECTIONS OF RIDGE FUNCTIONS

In practice, directions of image ridge functions and other structures are unknown. Given image f and n arbitrary projections $R_\omega f = (p_1,\dots,p_n)$ under directions $\omega = (\omega_1,\dots,\omega_n)$, we try to find n-tuple $\alpha = (\alpha_1,\dots,\alpha_n)$ of angles such that corresponding ridge functions $h_1,\dots,h_n$ compose a minimal norm solution H closest to function f. It was shown [7] that as a measure of closeness we can use the norm of minimal norm solution. Let us denote $H[\alpha|\omega](x,y)$ a minimal solution obtained from projections $R_\omega f$ by formulae (6) – (7) with the assumption that directions of ridge functions are from n-tuple α. Then

$$\left\| H[\alpha|\omega] \right\|^2 = \sum_{i=1}^{n} \int_{-1}^{1} h(\alpha_i, t)\, p(\omega_j, t)\, dt.$$

Hence the norm of minimal solution can be computed without backprojection procedure. Optimal set of angles $\alpha_{opt} = (\alpha_1,\dots,\alpha_n)$ can be found by consuming search of global maximum in n-dimensional volume:

$$\alpha_{opt} = \underset{\alpha \in [0,\pi)^n}{\arg\max} \left\| H[\alpha|\omega] \right\|.$$

Related results with detection of elongated bi-directional structures on digital images can be found in [9].

5. CONCLUSIONS

In these very first and preliminary experiments we try to check out applicability of the ridge functions in reconstruction of images containing elongated structures. Although, in general, due to the backprojection procedure, the filtered backprojection method can be considered also as a sum of some ridge (filtered and backprojected) functions, we should conclude that the standard technique of ρ-filtration is more reliable in comparison with expansion of projections into the Chebyshev series.

The suggested approach needs precise description of the range of its applicability. The experimental results show that in some cases of few-views tomography the use of ridge functions can extract information about structures even from nonsuitably directed projections. The approach can be of essential help in industrial tomography of objects which can be examined only within limited range of observation angles. Much work is needed for refinement of the algorithm and to automatically detect the optimal set of ridge functions and their directions.

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