Efficient Sampling of the Exponential Radon Transform

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Abstract *- In this short paper, we show that interlaced sampling can be used for the exponential radon Transform. Twice less data are needed for the same resolution in the reconstruction in tomography. This results can be generalized numerically to the exponential Radon transform, thus the attenuated Radon transform with a constant attenuation. This corresponds to emission tomography with a constant attenuation map*.

Keywords: Shannon, nuclear imaging, SPECT.

1. INTRODUCTION

The aim of industrial tomography is to reconstruct some physical parameters of a system (attenuation coefficient, nuclear activity, fluid velocity, etc.) from line integral measurements. The data are collected on sensors in a much more complex environment than in the medical field. Thus, few data is a common situation for industrial tomography. One of the crucial questions is to get the best information from the small amount of data.
A possible answer to the question of the best geometry of data acquisition can be given through the Shannon sampling theory. The interlaced sampling was first introduced in tomography by Cormack [1]. It was shown to be optimal in the framework of the Shannon theory by Rattey and Lindgren [2] and Natterer [3,4] for sampling the Radon transform $g(f,s)$ of f an essentially *b*-band limited function

$$g(f,s)=\int_{-\infty}^{+\infty} f(sq+tq^{\perp})dt,$$

with q is the unit vector $(\cos f,\sin f)$, $q^{\perp}$ is its orthogonal vector $(-\sin f,\cos f)$, $f\in[0,2p[$ is the angle of the projection (rotation parameter), s is the translation parameter and *b>0* is the band limit of the function f. The idea is to show that the Fourier transform $\hat{g}_k(s)$ of the function $g(f,s)$ (*k* is the index of the Fourier coefficient according to the first periodic variable) is negligible outside a set **K**. The function is sampled on schemes $Wl, l\in Z^2$ generated by a non singular sampling matrix *W* satisfying the Shannon non-overlapping condition of the sets $K+W^{-t}l, l\in Z^2$, where W^{-t} is the inverse of the transpose of *W*, see Figures 1 2 and 3 for illustrations in 2D tomography.

In [5], we showed that efficient sampling can be applied to the reconstruction of a catalyst powder in cracking plants. These results were then extended to vector field tomography [6] and recently to the 3D x-ray transform [7] and to 3D vector field tomography. Recent results in Doppler Imaging showed that interlaced sampling is also efficient for the generalized Radon transform with a polynomial weigh [8]

$$g(f,s)=\int_{-\infty}^{+\infty} f(sq+tq^{\perp})w(s,t)dt,$$

where $w(s,t)$ is a polynomial function in *s* and *t*.

The attenuated Radon transform appears naturally in nuclear imaging. For constant attenuation, the reconstruction can be reduced to the inversion of the exponential Radon transform, see [3]

$$\mathbf{T}_m f(q,s)=\int e^{mt} f(sq+tq^{\perp})dt,$$

where m >0 is the constant attenuation parameter and f denotes here the 2D-activity function to be reconstructed. The activity and the attenuation function are supposed to be of compact support, typically contained in the unit disk. The exponential weigh can be approximated by a polynomial in *t* on the compact support. Thus, from the sampling condition obtained on the generalized Radon transform with polynomial weigh, we can hope that interlaced schemes are efficient for the exponential Radon transform.

In this work, we show from numerical experiments that the interlaced geometry can be efficiently used for the reconstruction of nuclear activity in tomography. In the next section, we give a brief description of the nuclear waste control project in which this work will be applied in a next future, see also [9]. In the third section, we present numerical experiments showing that the interlaced sampling can be applied for the exponential Radon transform.

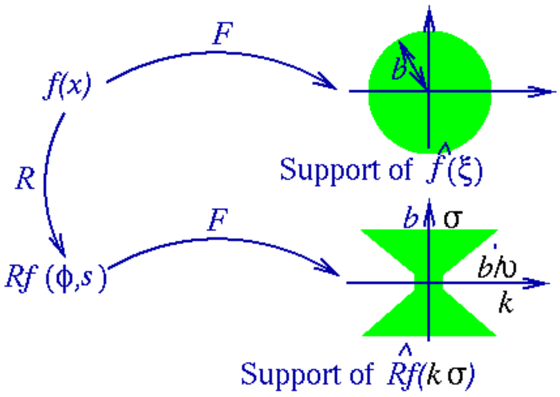

Figure 1 : Set K: Essential support of the Fourier transform of the 2D Radon transform of an essentially band limited function. Suppose that f is an essentially band-limited function, i.e., the essential support of its Fourier transform is a disk of radius b (more precisely, the integral of it's Fourier transform modulus is negligible outside a ball of radius b), then its Radon transform Rf is small outside the set K (drawn on the second line).

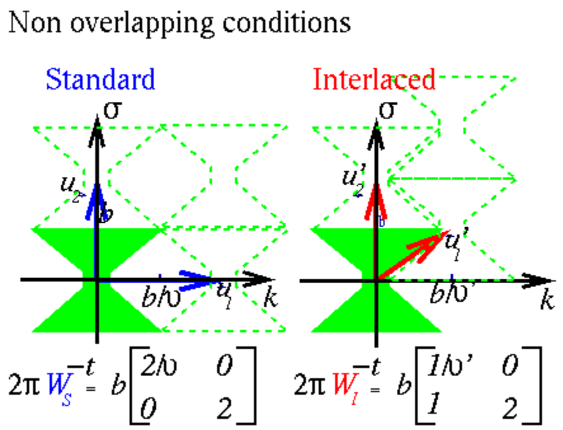

Figure 2 : Sampling condition of the 2D Radon transform. Standard schemes have equidistant steps in rotation and the translation. From the support of the 2D Fourier transform of the 2D Radon transform provided in the Figure 1, we se that the non-overlapping condition is not efficiently satisfied with the standard schemes (on the left). The sets $K + 2\pi W_I^{-t} l, l \in \mathbf{Z}^2$ satisfy more compactly the non-overlapping condition. This means that $\left|\det W_I^{-t}\right|$ is smaller, or equivalently that $\left|\det W_I\right|$ is larger and thus that fewer sampling points are needed to sample the function Rf .

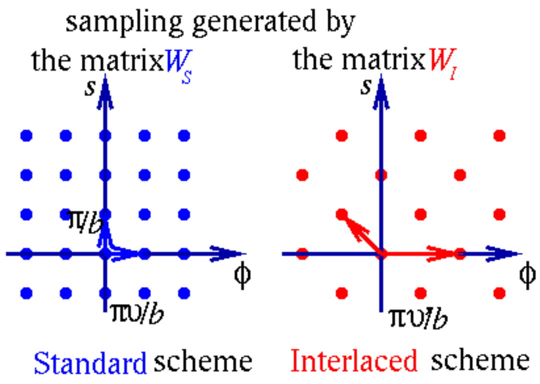

Figure 3 : Standard and interlaced sampling schemes. The best standard scheme satisfying the Shannon non overlapping condition has approximately twice more data than the interlaced scheme: indeed we have $\left|\det W_S\right| \approx 2\left|\det W_I\right|$. Interlaced schemes are thus more efficient in tomography.

2. CONTROL OF NUCLEAR WASTE PACKAGE ACTIVITY

Various nondestructive measuring techniques that use ionizing radiation are employed to characterize waste packages and raw waste. Total gamma-ray spectrometry is the most widely used nondestructive measuring technique because of its simple operation and low cost. Numerous laboratories have developed systems to measure gamma-emitting radionucleids in waste packages [10]. Their activity determination from gamma count needs some corrections depending on the apparatus and the physical characteristics of the drum considered as an active and spatially physically homogeneous source. The matrix homogeneity assumption is not always fulfilled. In that case, transmission tomography measurement can be matched to emission tomography in order to take into account the heterogeneity. For nuclear imaging, we use a parallel scanning geometry with only one detection channel. The acquisition time is function of the emergent activity and of the noise level supported by the reconstruction algorithm. We generally choose regularly spaced sampled projections on $[0,2\pi]$ and regularly spaced translations on the diameter of the cylindrical package, i.e., a standard scheme of *MN* measurements (*M* projections of *N* samples). Nevertheless, in order to cope with industrial demands, we work in a minimal sampling configuration. Typically the best sampling condition with FWHM=6.7 cm used to be *N*=9 (we generally choose *M*=9) for drum-radius of 28.5 cm [11]. Under these conditions, new efficient sampling schemes could improve the time process or/and image resolution.

3. SAMPLING THE EXPONENTIAL RADON TRANSFORM

In the following we present numerical experiments indicating that the interlaced scheme could be used for the exponential Radon transform, thus in nuclear imaging, if the assumption of constant attenuation is fulfilled. In Figure 4, we show the exponential radon transform of a hot spot in a strong attenuation medium. The number of angles on $[0,2p[$ is 320, the number of sample per-projection is 99. The attenuation domain is supposed to be the unit disk (normalization). In the unit disk, the attenuation coefficient m is 3. The hot spot is located in (0,-.5) on the vertical axis and is of radius .05 (build from the sum of 3 indicator functions of disk centered on (0,-.5) with respective radius .05, .02 and .01). The reconstruction using a classical filtered backprojection method shows artifacts due to the strong attenuation, see Figure 6. We show in Figure 5 the module of the 2D Discrete Fourier Transform (DFT) of the exponential Radon transform. The well-known bow tie shape appears. It contains the essential support of the Fourier transform of the data. Thus, interlaced sampling can be used in order to satisfy the Shannon non-overlapping condition, yielding a twice more efficient sampling scheme than the usual standard equidistant sampling.

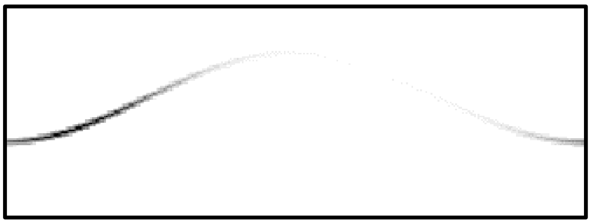

Figure 4: Sinogram. This is a standard sampling of the exponential Radon transform of a relative smooth pike. We present on this image the data. The horizontal axis is f ; the vertical axis is s .

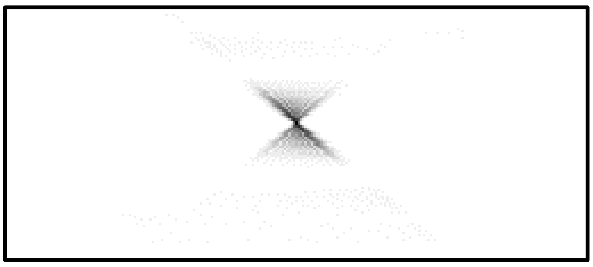

Figure 5: Modulus of the Fourier transform of the exponential Radon transform (see the data presented in Figure 4).

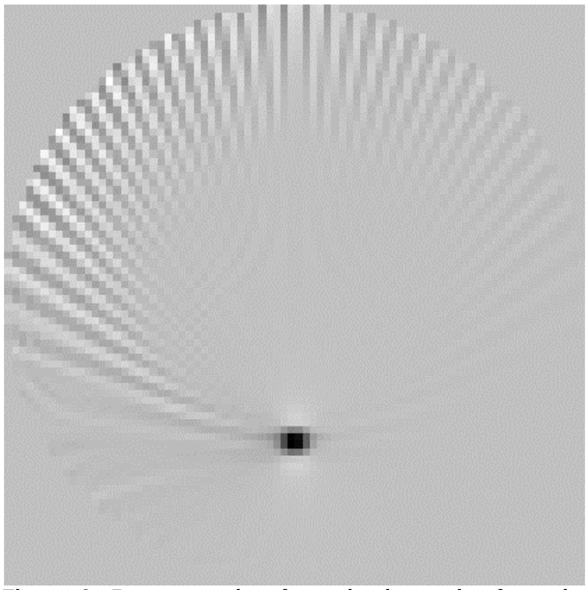

Figure 6: Reconstruction from the hot point from the exponential Radon transform. The reconstruction artifacts are due to the strong attenuation coefficient μ=3 (constant on the unit disk).

In Figures 7,8 and 9, we present a numerical experiment involving hot and cold activity areas in a middle activity background. The hot activity is 2. The cold activity is 0 and the background activity is 1. The support disk is as usual of unitary radius. The attenuation μ is 2. The data are given in Figure 7. The number of angles on $[0,2p[$ is 320, the number of sample per-projection is 99. The reconstruction is given in Figure 9. We show in Figure 8 the modulus of 2D Fourier transform of the data given in Figure 7. Again, the bow tie shape appears indicating that the interlaced sampling could be applied to sample this data set.

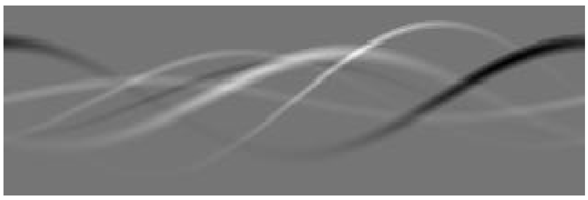

Figure 7: Sinogram. This is a standard sampling of the exponential Radon transform of a set of cold and hot areas in a constant active region. We present on this image the data. The horizontal axis is f ; the vertical axis is s .

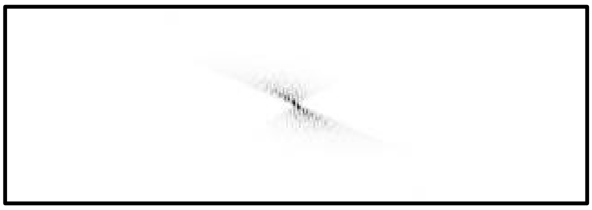

Figure 8: Modulus of the Fourier transform of the exponential Radon transform (see the data presented in Figure 7). We can recognize the bow tie shape indicating that interlaced sampling can be applied.

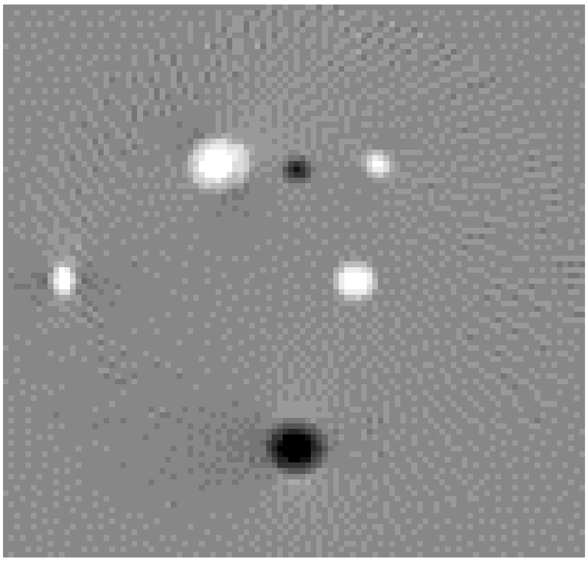

Figure 9: Reconstruction from the data presented in figure 7. The reconstruction does not suffer from strong artifacts. The attenuation coefficient is μ=2 (constant attenuation on the unit disk). It shows the cold activity region (black disks), the hot activity region (white disks) in a mean activity region.

4. DISCUSSION

The aim of this work is to study the sampling conditions in emission tomography. We have shown on two numerical examples that the interlaced scheme can be used in the context of constant attenuation. Our numerical experiments indicate that the interlaced scheme can be efficient for the exponential Radon transform. This result has to be confirmed by a rigorous mathematical proof and by several experiments on real data. The generalization to the attenuated Radon transform has to be studied so that this result can be used in emission tomography. The generalization to 3D has also to be provided. Efficient sampling could be used to sample the waste package activity in order to reduce the acquisition time. This work will be done in future months.

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