

Analytical Solution to the Three Dimensional Electrical Forward Problem for a Circular Cylinder

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Abstract- *In this paper, two techniques namely Mode Matching technique and Green's function technique are used to solve analytically the Forward Problem for a finite right circular cylinder. These techniques are compared in terms of analytical derivation and numerical results.*

Keywords : Electrical Impedance Tomography (EIT), Forward Problem, Green's function

1. INTRODUCTION

Electrical Resistivity Tomography (ERT)/ Electrical Impedance Tomography (EIT) is an emerging imaging technique with applications in the medical field and industrial process tomography (IPT). In the medical field, EIT has been used for the reconstruction of images of the human thorax [3]. In the industrial field, EIT has been used to monitor the flow of fluid through a pipeline and monitoring the mixing of two fluids [7]. In both fields there is increasing interest in improving the technique in order to provide quantitative information. In the medical domain the motivation for this is to improve diagnostic accuracy. This is also true for IPT applications, together with the additional realisation that interpreting qualitative images resulting from real time EIT systems to monitor fast industrial processes is untenable. However, characteristics of both domains present some formidable barriers to achieving this quantitative information.

In medicine, the situation is problematic due to the need to limit the energy of the imaging system and the need to contend with very complex boundary conditions and shapes (which may also be changing due to respiration etc.) In effect, this effectively limits such systems to differential imaging forms, although the advent of multi-frequency EIT systems allows the differential imaging to be performed both in the time in the frequency domains [4], [9]. Since the boundary condition of IPT applications are fixed to a large extent, the possibility of quantitative imaging arising from advanced image reconstruction algorithms exists. Both iterative image reconstruction algorithms and the more specialized forms such as layer-stripping require the accurate (and repeated) solution to the appropriate forward problem [5], [6]. In most situations the system being monitored or imaged

is three-dimensional and therefore the forward problem solver should reflect this.

Recently several groups have begun to address the three dimensional aspects of EIT [8], [10], [14], [16] with the aims of firstly reducing image distortions due to off plane conductivity changes associated with existing two dimensional image reconstruction algorithms and secondly, by increasing the measured boundary datasets, by placing electrodes over the entire surface of the body being imaged, potentially increasing the spatial resolution of the reconstructed images. This work is challenging in part due to the increased computational complexity compared to existing two dimensional image reconstruction algorithms for EIT.

To date most approaches aimed at addressing the three dimensional aspects of EIT involve solving the forward problem using approximation methods (such as Finite Element method or an infinite half-plane model to approximate 3D effects [14]) in order to avoid having to solve analytically the true forward problem.

In contrast, our work has focused on finding the analytical solution of the forward problem for three dimension problems for a precise geometrical figure (finite right circular cylinder) [10], [11]. We believe our work is directly relevant to ERT/EIT applications of IPT and our work is extendable to other areas such as Electrical Capacitance Tomography (ECT) [18]. The purpose of this paper is to present analytical expressions of the solution to forward problem for a finite right circular cylinder and to present some images of reconstructed equipotentials on the surface of the cylinder and inside of the cylinder when two electrodes are arbitrarily placed on its surface in order to inject a direct current.

2. FORWARD PROBLEM

A prerequisite in almost all EIT image reconstruction algorithms is access to the solution of the appropriate forward problem. A solution to the forward problem allows the potential arising from the application of current carrying electrodes onto a volume conductor to be found. Typically the repeated solution of the forward problem for a uniform conductivity distribution is required for the different drive configurations employed during data acquisition to allow the construction of a Sensitivity Matrix of the imaged object. Calculating the accurate solution to the forward problem and the construction of the Sensitivity Matrix for a three dimensional volume can be computationally very expensive. Once the Sensitivity Matrix is computed, it is typically then regularized and inverted which can be achieved using a variety of methods. The inverted Sensitivity Matrix is postmultiplied with the EIT dataset to reconstruct images. It is clear therefore that the image reconstruction depends both on the quality of the dataset and on the accuracy of the Sensitivity Matrix which itself depends on the accurate solution to the forward problem.

We have chosen to derive an analytical expression for the solution of the forward problem for a finite right circular cylinder on which two electrodes injecting a D.C. current are attached. This model approximates both medical imaging applications involving the human thorax and IPT applications involving pipes.

2.1 Solution to Laplace's equation using inhomogeneous boundary conditions

Since two electrodes are attached on the surface of the finite right circular cylinder and a direct current is injected into the system, an analytical solution to the forward problem for EIT requires solving Laplace's equation using inhomogeneous Neumann boundary conditions.

The geometry of the finite circular cylinder with two electrodes attached on its surface is shown in figure 1.

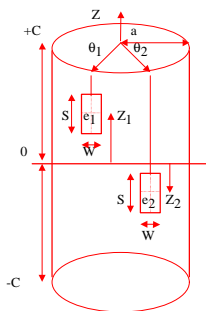


Figure 1. Parameters of the cylinder showing electrodes placement

A solution to Laplace's equation is required :

$$\nabla^2 \Phi = 0 \tag{1}$$

with Φ being the potential,
 ∇^2 being the Laplacian operator,

with inhomogeneous boundary conditions given by :

$$\frac{\partial \Phi}{\partial n} = 0 \left\{ \begin{array}{l} \text{on the boundary of the cylinder} \\ \text{which does not include the electrodes,} \end{array} \right.$$

$$S \frac{\partial \Phi}{\partial n} = J \left\{ \begin{array}{l} \text{on only the boundary of the cylinder which} \\ \text{includes the electrodes,} \end{array} \right. \tag{2}$$

where J is the current density, σ is the conductivity and $\partial/\partial n$ is the normal derivative to the surface.

A solution to Laplace's equation is found by solving equation (1) expressed in cylindrical coordinates as follow

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{3}$$

with the boundary conditions described above.

The full solution to Laplace's equation using these boundary conditions [13] is given by :

$$\Phi(r, \theta, z) = \frac{-4Ic}{s p^3 \Delta s (1 + d_{0n})} \sum_{n=0}^{\infty} \frac{1}{n} \left[\sum_{r=1}^{\infty} \frac{1}{r^2} \sin\left(\frac{r p z}{2c}\right) \left[\cos\left(\frac{r p z_1}{c}\right) \cos(n(\theta - \theta_1)) - \cos\left(\frac{r p z_2}{c}\right) \cos(n(\theta - \theta_2)) \right] \right. \\ \left. \times \frac{I_n\left(\frac{r p r}{c}\right)}{I_n'\left(\frac{r p a}{c}\right)} \cos\left(\frac{r p z}{c}\right) \right] \sin\left(\frac{n \Delta}{2a}\right) - \\ \frac{4Ic}{s p^3 \Delta s (1 + d_{0n})} \sum_{n=0}^{\infty} \frac{1}{n} \left[\sum_{r=1}^{\infty} \frac{1}{r^2} \sin\left(\frac{(2r+1) p z}{4c}\right) \left[\sin\left(\frac{(2r+1) p z_1}{2c}\right) \cos(n(\theta - \theta_1)) \right. \right. \\ \left. \left. - \sin\left(\frac{(2r+1) p z_2}{2c}\right) \cos(n(\theta - \theta_2)) \right] \right] \frac{I_n\left(\frac{(2r+1) p r}{2c}\right)}{I_n'\left(\frac{(2r+1) p a}{2c}\right)} \sin\left(\frac{(2r+1) p z}{2c}\right) \sin\left(\frac{n \Delta}{2a}\right) - \\ \frac{Ia}{s p^3 \Delta c} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{r}{a}\right)^n \sin\left(\frac{n \Delta}{2a}\right) \left[\cos(n(\theta - \theta_1)) - \cos(n(\theta - \theta_2)) \right].$$

where d_{0n} is the Kronecker delta symbol

$$-d_{0n} = 1 \text{ when } n = 0$$

$$-d_{0n} = 0 \text{ when } n \neq 0$$

I being the current injected,

$$s : \text{conductivity}, \Delta = \frac{W}{2a}$$

I_n is the modified Bessel function of first kind,

I_n' is the derivative of modified Bessel function of first kind,

Whilst, the above expression allows the forward problem arising from bipolar drive electrodes placed arbitrarily on the curved surface of the cylinder to be found, its

implementation and resulting numerical behaviour are not trivial matters. Indeed on closer inspection, this expression reveals some important limitations. Firstly, this expression involves the use of only two rectangular electrodes. Secondly, it involves the use of modified Bessel functions which give rise to problems when the potential is computed on the boundary. Nevertheless, this expression was the first expression presented for solving Laplace's equation for a finite right circular cylinder with two rectangular electrodes on its surface and we were able to reconstruct an image of an object using a Sensitivity Matrix calculated using this expression [10].

In order to overcome the above limitations, we have been exploring alternative analytical solutions to the same forward problem which may: be easier to implement, have a better computational behaviors and be extendable to N electrode drive configurations.

The next section shows how these limitations can be overcome.

2.2 Solution to Poisson's equation using homogeneous boundary conditions

An alternative formulation to the forward problem is to consider replacing the electrodes located on the boundary with electrical monopoles located within the cylinder [17] thereby allowing Laplace's equation to be replaced with Poisson's equation and thereby transforming the inhomogeneous Neumann boundary conditions (given by equation (2)) into homogeneous Neumann boundary conditions.

Poisson's equation is given by :

$$\nabla^2 \Phi = -J(r_0) \tag{5}$$

where $J(r_0)$ is the current source at an internal point (r_0) .

Equation (5) can be solved by the application of the Green's function. For a solution to equation (5), the source needs to be replaced by an impulse function. Therefore, equation (5) can be rewritten as

$$\nabla^2 G(r|r_0) = -\mathbf{d}(r-r_0) \tag{6}$$

where $G(r|r_0)$ is the Green's function,

r_0 is the actual position of the current source,

r is the location at which the solution is to be found.

The Green's function in equation (6) can be obtained by using the eigenfunction expansion technique [15]. In this technique, the solution is first expressed as an infinite series of

eigenfunctions of the required Poisson's equation. Then substituting back into equation (6) and using the orthogonality of the eigenfunctions, the modal coefficients are determined. Once the modal coefficients are known, the Green's function can be found. Having found the Green's function from equation (6), the solution (Φ) in (5) can be found by multiplying the Green's function with the current source given in (5) and integrating the result of this multiplication over the surface of each electrode. Then the total solution (Φ) at position (r) is found by summing the result of each integration over the electrodes, as follows :

$$\Phi(r) = - \sum_{i=1}^{\text{number of electrodes}} \iint_A G(r|r_0) J_i(r_0) dr_0 \tag{7}$$

where J_i is the current density applied on the i^{th} electrode,

A is the surface area of the i^{th} electrode.

The full solution to Poisson's equation using homogeneous boundary conditions can be shown [12] to be :

$$\begin{aligned} \Phi(r, \mathbf{f}, z) = & - \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} \frac{\mathbf{s}_n \mathbf{s}_r 2a \Delta}{c p \mathbf{b}_r} \frac{1}{(a^2 k_{nm}^2 - n^2)} \frac{J_n(k_{nm} r)}{J_n(k_{nm} a)} \\ & \times \frac{k_{nm}^2}{k_{nm}^2 + \mathbf{b}_r^2} \text{sinc}(n\Delta) \sin(\mathbf{b}_r \frac{S}{2}) \\ & \times [\cos(n(\mathbf{f} - \mathbf{f}_1)) \cos(\mathbf{b}_r(z_1 - c)) \\ & - \cos(n(\mathbf{f} - \mathbf{f}_2)) \cos(\mathbf{b}_r(z_2 - c))] \\ & \times \cos(\mathbf{b}_r(z - c)) \\ & - \sum_{r=1}^{\infty} \frac{4J\Delta}{c p \mathbf{b}_r^3 a} \sin(\mathbf{b}_r \frac{S}{2}) \cos(\mathbf{b}_r(z - c)) \\ & [\cos(\mathbf{b}_r(z_1 - c)) - \cos(\mathbf{b}_r(z_2 - c))] \tag{8} \end{aligned}$$

where $J = \frac{I}{\mathbf{s}SW}$ with I being the current injected,

\mathbf{s} being the conductivity,

$$\Delta = \frac{W}{2a}, \text{sinc}(n\Delta) = \frac{\sin(n\Delta)}{n\Delta}, \mathbf{b}_r = \frac{r\mathbf{p}}{2c},$$

J_n is the n^{th} Bessel function of first kind,

k_{nm} is the m^{th} root of $J'_n(k_{nm} a) = 0$,

$$\mathbf{s}_n = \begin{cases} 1 & n = 0 \\ 2 & n \neq 0 \end{cases}, \mathbf{s}_r = \begin{cases} 1 & r = 0 \\ 2 & r \neq 0. \end{cases}$$

3. RESULTS

3.1 Analytical results

A solution to Laplace's equation for a finite right circular cylinder using rectangular electrodes and satisfying the boundary conditions described above is presented in [12]. However, the resulting equation is complex and not easily implemented. For situations where electrodes of the same size are employed the expression can be simplified to:

$$\Phi(\mathbf{r}, \mathbf{f}, z) = - \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} \frac{2J_s s_n}{nc \rho \mathbf{b}_r^2} \frac{I_n(\mathbf{b}_r \mathbf{r})}{I_n'(\mathbf{b}_r a)} \sin(n\Delta) \sin(\mathbf{b}_r \frac{S}{2}) \times [\cos(n(\mathbf{f} - \mathbf{f}_1)) \cos(\mathbf{b}_r(z_1 - c)) - \cos(n(\mathbf{f} - \mathbf{f}_2)) \cos(\mathbf{b}_r(z_2 - c))] \times \cos(\mathbf{b}_r(z - c)) - \sum_{n=1}^{\infty} \frac{SaJ}{\rho c} \left(\frac{\mathbf{r}}{a}\right)^n \frac{1}{n^2} \sin(n\Delta) [\cos(n(\mathbf{f} - \mathbf{f}_1)) - \cos(n(\mathbf{f} - \mathbf{f}_2))] \quad (9)$$

where $J = \frac{I}{sSW}$ with I being the current injected,

$$s : \text{conductivity}, \Delta = \frac{W}{2a}, \mathbf{b}_r = \frac{r\rho}{2c},$$

I_n is the modified Bessel function of first kind,

I_n' is the derivative of modified Bessel function of first kind,

$$s_n = \begin{cases} 1 & n = 0 \\ 2 & n \neq 0. \end{cases}$$

It is interesting to note that equation (8) can be reduced to equation (9) by applying the Mittag-Leffler's theorem [1] when identical electrodes used in bipolar drive configurations are employed. In other words, Mittag-Leffler's theorem transforms equation (8) into equation (9) which is a reduced form of the solution given in [13] to Laplace's equation with inhomogeneous Neumann boundary conditions for a finite right circular cylinder of homogeneous isotropic conductivity on which two rectangular electrodes of the same size are attached injecting a direct current.

3.2 Experimental results

Firstly, the reduced form given by equation (8) is compared with the original form (equation(4)) in terms of reconstruction of the equipotentials on the boundary and within the XY plane at $z=0$. The cylinder had a height of 2.7cm and a radius of 0.715cm. The two rectangular electrodes had a height of 0.26cm and a width of 0.14cm. Electrodes were attached on the boundary of the cylinder in an asymmetric configuration. The first electrode was positioned at 200 degrees at $z=1.0$ cm. The second electrodes was positioned at 35 degrees at $z=0.0$ cm. The conductivity was uniform (1 S/m) and the applied current was 27.847 mA. For both expressions, the reconstructed equipotentials have the same shape. Therefore, only the reconstructed equipotentials for equation (9) are presented in this paper. Figure 2 shows the reconstructed equipotentials along the z axis on the boundary of the cylinder. Figure 3 shows the reconstructed equipotentials in the xy plane at $z=0$ for electrode 1.

Secondly, the reduced form is compared with the original form (equation (4)) in terms of potentials calculated on five points arbitrarily selected situated both on the boundary and inside of the cylinder but all in the same plane ($z=0$). Table 1 shows the results for these five points for the original form presented in [12] and the reduced form given by equation (8). Similar correspondence between calculated potentials using the two expressions was obtained for other points including those not in the $z=0$ plane.

4. DISCUSSION

Figure 2 and 3 shows that the reconstructed equipotentials are well behaved and have the expected shape.

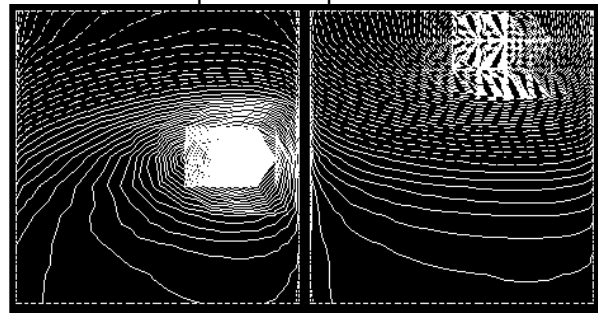


Figure 2. Reconstruction of the equipotentials for the asymmetric drive configuration on the boundary.

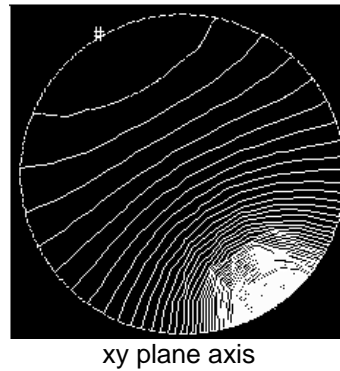


Figure 3. Reconstruction of the equipotentials for the asymmetric drive configuration in the xy plane where one of the electrode is located (electrode 1).

Table 1 shows that the reduced form has the same potential values as the original form up to seven significant figures. After seven significant figures the result from the two expressions start to differ. This is due to the different computational implementations of the two expressions.

Point	Original Form	Reduced Form
0.0 0.715	-4.3434232	-4.3434231
-0.1395 0.7013	-3.5931180	-3.5931179
-0.2736 0.6606	-2.9376768	-2.9376766
-0.3972 0.5945	-2.3662183	-2.3662182
-0.5056 0.5056	-1.8639928	-1.8639927

Table 1. Comparison of potentials (mV) calculated for five points situated both on the boundary and inside of the cylinder but in the same plane ($z=0$) for the original (equation(4)) and the reduced form shown in equation (8).

4. CONCLUSIONS

In this paper, two solutions to the forward problem are presented for a finite right circular cylinder on whose curved surface two rectangular electrodes are placed arbitrarily for injecting a direct current. The first solution given by [13] corresponds to the solution to Laplace's equation with inhomogeneous Neumann boundary conditions. This expression can be reduced to a new form when two identical electrodes are employed. The second solution is based on Poisson's equation using homogeneous Neumann boundary conditions. This expression can be transformed to the reduced form justification for employing simpler forward problem of the first solution by the use of the Mittag-Leffler's theorem.

The above is an interesting result and presents some solutions for EIT applications, as already employed by some [2], [17]. However, some caveats to the above findings should be noted. Firstly, the relationship between Laplace's equation and Poisson's equation exists only because the two rectangular electrodes have the same size and inject a direct current. Secondly, this relationship exists because the source in the Poisson's equation is very close to the boundary. Further work is required to ascertain if the equivalence between Laplace's and Poisson's results can be generalized to different sized electrodes, non-bipolar drive configurations and different geometrical bodies. Finally, the Green's function technique as used here in solving Poisson's equation using homogeneous boundary conditions for a finite right circular cylinder presents three advantages over the Mode Matching technique used in [13] to solve Laplace's equation. Firstly, it provides two forms of the solution to Poisson's equation given by equation (8) and (9). Secondly, it provides directly the reduced form given by equation (9) of the solution given in [13]. And thirdly, the first form given by equation (8) uses Bessel functions which are more stable than the modified Bessel functions used in equation (9) and in [13] and therefore provides a more stable computational implementation [12].

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